Fractional Calculus & Applied Analysis
An International Journal for Theory and Applications
VOLUME 11, NUMBER 3 (2008) ISSN 1311-0454

ON THE SEPARATION OF SOLUTIONS OF FRACTIONAL DIFFERENTIAL EQUATIONS

Kai Diethelm

Dedicated to Professor Dr. Paul L. Butzer on the occasion of his 80th birthday

Abstract

Consider two different solutions of a first-order differential equation. Under rather general conditions we know that these two functions are separated from each other, i.e. their graphs never meet over even cross each other. We ask whether such a result is true for Caputo-type fractional differential equations as well. We can give a partial answer that is positive in some situations and negative under different assumptions. For the remaining cases we state a conjecture and explain why we believe in it. A key ingredient of the analysis is a result concerning the existence of zeros of the solutions of a certain class of Volterra equations.

Mathematics Subject Classification: 34C10 (primary); 26A33, 34A26, 45D05, 45E10 (secondary)

Key Words and Phrases: Caputo derivative; fractional differential equation; separation of solutions; weakly singular Volterra integral equation; existence of zeros

1. Introduction and formulation of the problem

One of the most fundamental and best known results in the theory of classical ordinary differential equations deals with the question whether the graphs of two different solutions of the same differential equation can meet or even cross each other. Under quite natural assumptions, the answer is negative, i.e. the graphs are strictly separated from each other. A mathematically precise formulation reads as follows (see, e.g., [3, Thm. 3.1]):

THEOREM 1.1. *Let $f : [a,b] \times [c,d] \to \mathbb{R}$ be continuous and satisfy a Lipschitz condition with respect to the second variable. Consider two solutions y_1 and y_2 of the differential equation*

$$y_j'(x) = f(x, y_j(x)) \qquad (j = 1, 2) \tag{1}$$

subject to the initial conditions $y_j(x_{j0}) = y_{j0} \in (c,d)$, respectively. Then the functions y_1 and y_2 coincide either everywhere or nowhere.

P r o o f. The proof is very simple: Assume that y_1 and y_2 coincide at some point x^*, i.e. $y_1(x^*) = y_2(x^*) =: y^*$, say. Then, both functions solve the initial value problem $y_j'(x) = f(x, y_j(x))$, $y_j(x^*) = y^*$. Since the assumptions assert that this problem has a unique solution, y_1 and y_2 must be identical, i.e. they coincide everywhere. ■

The graphs of two different solutions of eq. (1) thus never meet or cross each other. This result can be seen as the basis of graphical methods for solving first-order differential equations in the sense that it allows to plot the graph of a solution on the basis of a direction field as indicated in the example $y'(x) = x - y(x)$ in Figure 1. If the graphs of two solutions would meet then the direction field would not give any useful information.

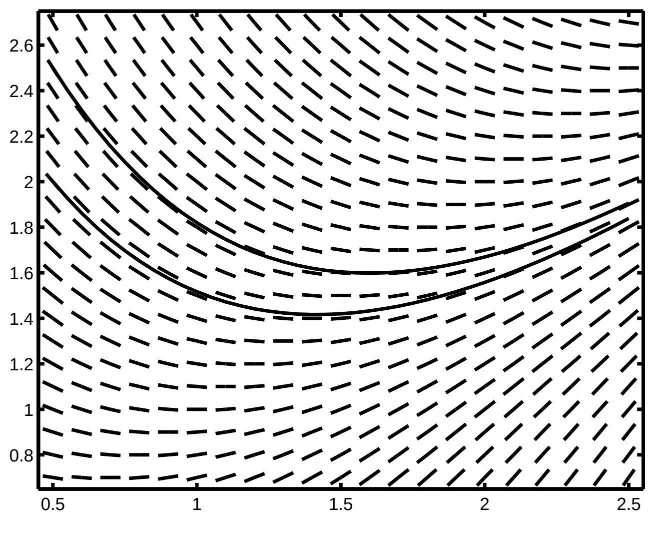

Figure 1: Direction field for $y'(x) = x - y(x)$ and two solutions.

We specifically draw the reader's attention to the fact that in Theorem 1.1 it does not matter whether the values x_1 and x_2, i.e. the abscissas where the initial conditions are specified, coincide with each other or not.

We now try to generalize this result to the fractional setting, i.e. to differential equations of the form

$$D^{\alpha}_{*a} y(x) = f(x, y(x)), \tag{2}$$

where D^{α}_{*a} denotes the Caputo differential operator of order $\alpha \notin \mathbb{N}$ (see [2]), defined by

$$D^{\alpha}_{*a} y(x) := D^{\alpha}_{a}(y - T[y])(x),$$

where $T[y]$ is the Taylor polynomial of degree $\lfloor \alpha \rfloor$ for y, centered at a, and D^{α}_{a} is the Riemann-Liouville derivative of order α [6]. The latter is defined by $D^{\alpha} := D^{\lceil \alpha \rceil} J_a^{\lceil \alpha \rceil - \alpha}$, with J_a^{β} being the Riemann-Liouville integral operator,

$$J_a^{\beta} y(x) := \frac{1}{\Gamma(\beta)} \int_a^x (x-t)^{\beta-1} y(t) dt.$$

When attempting this generalization, one first notices that the proof of Theorem 1.1 strongly depends on the locality of the differential operator of order 1. Such a property is not available in the fractional case, and hence the proof cannot be carried over directly. Moreover, as a consequence of this non-locality, we find that now it does matter whether the abscissas of the initial conditions coincide or not. Indeed the following example shows that we cannot expect a result comparable to Theorem 1.1 to hold if $x_1 \neq x_2$:

Example 1.1. Let $0 < \alpha < 1$ and consider the fractional differential equations

$$D^{\alpha}_{*0} y_1(x) = \Gamma(\alpha + 1), \qquad y_1(0) = 0,$$

and

$$D^{\alpha}_{*1} y_2(x) = \Gamma(\alpha + 1), \qquad y_2(1) = 1.$$

We have here two differential equations with identical right-hand sides but initial conditions specified at different points. The solutions are easily seen to be $y_1(x) = x^{\alpha}$ and $y_2(x) = 1 + (x-1)^{\alpha}$. It is obvious that these two functions coincide at $x = 1$ but nowhere else.

Theorem 1.1 deals with differential equations of order 1, i.e. with equations subject to exactly one initial condition. It is well known that a similar statement does not hold for equations of higher order (for example, the equation $y''(x) = -y(x)$ has solutions $y(x) = \cos x$, $y(x) = \sin x$ and $y(x) = 0$ the first two of which oscillate and the graphs of which cross each other).

Similar effects arise in the context of fractional differential equations with more than one initial condition (i.e. equations of order $\alpha > 1$); see [5]. Therefore all we can hope for is summarized in the following conjecture.

Conjecture 1.2. *Let $0 < \alpha < 1$ and assume $f : [0, b] \times [c, d] \to \mathbb{R}$ to be continuous and satisfy a Lipschitz condition with respect to the second variable. Consider two solutions y_1 and y_2 of the differential equation*

$$D^{\alpha}_{*0} y_j(x) = f(x, y_j(x)) \qquad (j = 1, 2) \tag{3}$$

subject to the initial conditions $y_j(0) = y_{j0}$, respectively, where $y_{10} \neq y_{20}$. Then, for all x where both $y_1(x)$ and $y_2(x)$ exist, we have $y_1(x) \neq y_2(x)$.

We have calculated solutions for a number of special cases and the results made us believe that the conjecture is true even though at present we are not aware of a proof under these general conditions. However we can prove the conjecture at least for a restricted range of α under certain additional conditions; see Section 2 below. To this end we provide an equivalent formulation that may be easier to handle.

Conjecture 1.3. *Let $0 < \alpha < 1$ and assume $g : [0, b] \times [y^* - K, y^* + K] \to \mathbb{R}$ to be continuous and satisfy a Lipschitz condition with respect to the second variable. Moreover, let $g(x, 0) = 0$ for all $x \in [a, b]$, and let $y^* \neq 0$. Then, the solution y of the Volterra equation*

$$y(x) = y^* + \frac{1}{\Gamma(\alpha)} \int_0^x (x - t)^{\alpha - 1} g(t, y(t)) dt \tag{4}$$

satisfies $y(x) \neq 0$ for all x.

It is an immediate consequence of Theorem 1.1 that this conjecture is true for $\alpha = 1$. Moreover, we shall see in Theorem 2.1 below that it also holds for α very close to zero under mild additional conditions on the given function g. Furthermore we know that the conjecture is true for all $\alpha \in (0, 1)$ in the case of a linear equation with constant coefficients, i.e. if $g(x, y) = cy$ with some constant c [5, §4]. These facts support our belief in the truth of this conjecture for all $\alpha \in (0, 1)$. If $\alpha > 1$ however then we must not expect such a result to hold. This follows from the counterexample $g(x, y) = -y$ that is discussed extensively in [5].

The relation between these two conjectures is simple:

Theorem 1.4. *Conjecture 1.2 holds if and only if Conjecture 1.3 holds.*

P r o o f. In this proof we follow the lines of [3, p. 89]. We first show that Conjecture 1.2 implies Conjecture 1.3. To this end we recall that eq. (4) is equivalent to the fractional initial value problem

$$D^{\alpha}_{*0} y(x) = g(x, y(x)), \qquad y(0) = y^* \neq 0;$$

see, e.g., [4, Lemma 2.1]. Our assumption that $g(x, 0) = 0$ for all x implies that the function $z(x) = 0$ solves the initial value problem

$$D^{\alpha}_{*0} z(x) = g(x, z(x)), \qquad z(0) = 0.$$

Thus, if Conjecture 1.2 is true we may conclude that $y(x) \neq z(x) = 0$ for all x. Conjecture 1.3 is therefore true as well.

For the other direction, let y_1 and y_2 be the solutions of the initial value problems mentioned in Conjecture 1.2, and define

$$g(x, z) := f(x, z + y_1(x)) - f(x, y_1(x)).$$

Moreover we set $y(x) := y_2(x) - y_1(x)$ and $y^* := y_{20} - y_{10} \neq 0$. Now we rewrite the two initial value problems from Conjecture 1.2 in their corresponding Volterra forms, viz.

$$y_j(x) = y_{j0} + \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t, y_j(t)) dt \qquad (j = 1, 2).$$

Subtracting these two equations, we find

$$\begin{aligned} y(x) &= y_2(x) - y_1(x) \\ &= y_{20} - y_{10} + \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} [f(t, y_2(t)) - f(t, y_1(t))] dt \\ &= y^* + \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} g(t, y_2(t) - y_1(t)) dt \\ &= y^* + \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} g(t, y(t)) dt. \end{aligned}$$

Thus, y solves the Volterra equation (4). Since $g(x, 0) = f(x, y_1(x)) - f(x, y_1(x)) = 0$ for all x, we may now apply Conjecture 1.3 to conclude that $0 \neq y(x) = y_2(x) - y_1(x)$ for all x. ∎

2. Partial solutions to the problems

We can actually prove the following weaker version of Conjecture 1.3.

THEOREM 2.1. *Consider the solution y of the Volterra equation*

$$y(x) = y^* + \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} g(t, y(t)) dt \tag{5}$$

subject to the following conditions:

(a) $g : [0, b] \times [y^* - K, y^* + K] \to \mathbb{R}$ *is continuous and satisfies a Lipschitz condition with Lipschitz constant L with respect to the second variable.*

(b) $y^* \neq 0$.

(c) $g(x, 0) = 0$ *for all* $x \in [0, b]$.

(d) There exist two constants $c_1, c_2 > 0$ such that $c_1 + c_2 < 1$ and $c_2|y^*| \geq L\Omega$ where Ω is a positive constant that satisfies $\omega(y; h) \leq \Omega h^\alpha$ for all h, with $\omega(y; \cdot)$ denoting the modulus of continuity of y.

Then, $y(x) \neq 0$ for all x for which the solution exists if $0 < \alpha < \alpha_0$ with some constant α_0.

The same statement holds if condition (d) is replaced by

(e) $g \in C^1([0, b] \times [y^* - K, y^* + K])$.

REMARK 2.1. Before coming to the proof, let us comment on the constant Ω whose existence is required in condition (d). For a fixed α we know from [4, Theorem 2.2] that the solution y of eq. (5) is continuous. Then, the function z with $z(x) := g(x, y(x))$ is continuous too. By eq. (5), $y(x) = y^* + J_0^\alpha z(x)$, and the right-hand side of this equation is in the Hölder space H_α because of [6, Theorem 3.1]. By definition of this Hölder space we obtain the existence of the required value Ω. However, this value depends on the solution y and thus indirectly on α. It is not clear a priori how Ω behaves as α varies. Our theorem requires an Ω satisfying condition (d) for all $\alpha \in (0, \alpha_0)$. Thus what we actually need is that Ω remains bounded as $\alpha \to 0$. Whether or not this holds depends on the given function g.

REMARK 2.2. Another remark is in order with respect to condition (e). If only conditions (a), (b), and (c) were satisfied then one might be tempted to approximate the given function g by the elements of a sequence (g_n) with the properties that (i) $g_n \in C^1([0, b] \times [y^* - K, y^* + K])$, and (ii)

$\lim_{n\to\infty} \|g - g_n\|_\infty = 0$. It is well known that such a sequence exists and that the solutions y_n of the equations

$$y_n(x) = y^* + \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} g_n(t, y_n(t))dt$$

satisfy $y_n \to y$ uniformly [4, Theorem 3.4]. Moreover, for each n we may apply Theorem 2.1. This gives us an α_0 for each n. We may then say that the solution y of the originally given equation does not have a zero whenever $0 < \alpha < \lim_{n\to\infty} \alpha_0$. Of course, this information is useful only if $\lim_{n\to\infty} \alpha_0 > 0$. Whether or not this holds once again depends on g and probably also on the sequence (g_n).

P r o o f. Let us assume that $y(x^*) = 0$ for some $x^* > 0$. We first prove the claim under assumptions (a)–(d). To this end we define the quantities

$$M_1(\alpha) := \left(\frac{x^* \|g\|_\infty}{c_1 |y^*| \Gamma(\alpha)}\right)^{1/(1-\alpha)} \quad \text{and } M_2(\alpha) := \left(\frac{c_2 |y^*| \Gamma(\alpha+1)}{\Omega L}\right)^{1/(2\alpha)}$$

and note that, under our conditions,

$$M_1(\alpha) = O(\alpha) \to 0 \quad \text{and} \quad M_2(\alpha) \to \infty \quad \text{as } \alpha \to 0.$$

Thus we may find some $\alpha_0 > 0$ such that $M_1(\alpha) \le M_2(\alpha)$ for all $\alpha \in (0, \alpha_0)$. Let us now choose an arbitrary α is in this range and let $\epsilon^* \in [M_1(\alpha), M_2(\alpha)]$ and $\epsilon := \min\{x^*, \epsilon^*\}$. Then, we rewrite the Volterra equation (5) at the point $x = x^*$ in the form

$$0 = y(x^*) = y^* + I_1 + I_2\,, \tag{6}$$

where

$$I_1 := \frac{1}{\Gamma(\alpha)} \int_0^{x^*-\epsilon} (x^*-t)^{\alpha-1} g(t, y(t))dt$$

and

$$I_2 := \frac{1}{\Gamma(\alpha)} \int_{x^*-\epsilon}^{x^*} (x^*-t)^{\alpha-1} g(t, y(t))dt.$$

Now we can estimate I_1 and I_2. Firstly, the Mean Value Theorem for integrals implies that we may find $\xi \in (x^* - \epsilon, x^*)$ such that

$$\begin{aligned}
|I_2| &= |g(\xi, y(\xi))| \frac{1}{\Gamma(\alpha)} \int_{x^*-\epsilon}^{x^*} (x^*-t)^{\alpha-1} dt \\
&= |g(\xi, y(\xi)) - g(\xi, y(x^*))| \frac{1}{\Gamma(\alpha+1)} \epsilon^\alpha \le L|y(\xi) - y(x^*)| \frac{1}{\Gamma(\alpha+1)} \epsilon^\alpha \\
&\le \frac{L\Omega}{\Gamma(\alpha+1)} \epsilon^{2\alpha} \le \frac{L\Omega}{\Gamma(\alpha+1)} [\epsilon^*]^{2\alpha} \le \frac{L\Omega}{\Gamma(\alpha+1)} [M_2(\alpha)]^{2\alpha} = c_2|y^*|.
\end{aligned}$$

Moreover, if $\epsilon^* > x^*$ then $\epsilon = x^*$ and hence $I_1 = 0$. Otherwise, $\epsilon = \epsilon^*$ and thus

$$\begin{aligned}
|I_1| &\leq \frac{1}{\Gamma(\alpha)} \int_\epsilon^{x^*} t^{\alpha-1} |g(x^* - t, y(x^* - t))| dt \\
&\leq \frac{\epsilon^{\alpha-1}}{\Gamma(\alpha)} \int_\epsilon^{x^*} |g(x^* - t, y(x^* - t))| dt \\
&\leq \frac{\epsilon^{\alpha-1}}{\Gamma(\alpha)} \int_0^{x^*} |g(x^* - t, y(x^* - t))| dt \leq \frac{\epsilon^{\alpha-1} x^* \|g\|_\infty}{\Gamma(\alpha)} \\
&= \frac{[\epsilon^*]^{\alpha-1} x^* \|g\|_\infty}{\Gamma(\alpha)} \leq \frac{[M_1(\alpha)]^{\alpha-1} x^* \|g\|_\infty}{\Gamma(\alpha)} = c_1 |y^*|.
\end{aligned}$$

We conclude that

$$|I_1 + I_2| \leq |I_1| + |I_2| \leq (c_1 + c_2)|y^*| < |y^*|.$$

On the other hand, eq. (6) implies $I_1 + I_2 = -y^*$, i.e.

$$|I_1 + I_2| = |y^*|$$

and we obtain the desired contradiction. Thus, our assumption that some x^* exists with $y(x^*) = 0$ must be false.

In the case of the hypotheses (a), (b), (c) and (e), we proceed in a similar way. First of all we know from [1, Theorem 2.1] that assumption (e) implies that $y \in C^1(0, x^*]$. Hence, $\Omega := \sup_{x^*/2 \leq x \leq x^*} |y'(x)|$ is a finite number. We then choose two positive numbers c_1 and c_2 such that $c_1 + c_2 < 1$, define $M_1(\alpha)$ as above, and set

$$M_2(\alpha) := \left(\frac{c_2 |y^*| \Gamma(\alpha + 1)}{\Omega L} \right)^{1/(\alpha+1)}.$$

It then turns out that $M_2(\alpha) \to c_2|y^*|/(\Omega L) > 0$ as $\alpha \to 0$. Thus we may once again conclude that there exists some $\epsilon^* \in [M_1(\alpha), M_2(\alpha)]$ whenever $0 < \alpha < \alpha_1$ for some suitable $\alpha_1 > 0$. Moreover it is evident that

$$\left(\frac{x^*}{2} \right)^\alpha \|g\|_\infty \leq c_1 |y^*| \Gamma(\alpha)$$

for $0 < \alpha < \alpha_2$ with some $\alpha_2 > 0$ since the right-hand side tends to ∞ as $\alpha \to 0$ whereas the left-hand side remains bounded. For $\alpha_0 := \min\{\alpha_1, \alpha_2\}$ we then choose an arbitrary $\alpha \in (0, \alpha_0)$ and set $\epsilon := \min\{x^*/2, \epsilon^*\}$. Then we may estimate I_1 as above if $\epsilon^* \leq x^*/2$ and see that $I_1 \leq c_1|y^*|$. In the other case, we proceed by writing

$$\begin{aligned}|I_1| &\leq \frac{1}{\Gamma(\alpha)}\int_{x^*/2}^{x^*} t^{\alpha-1}|g(x^*-t, y(x^*-t))|dt \\ &\leq \frac{(x^*/2)^{\alpha-1}}{\Gamma(\alpha)}\int_{x^*/2}^{x^*}|g(x^*-t, y(x^*-t))|dt \leq \frac{(x^*/2)^{\alpha}\|g\|_\infty}{\Gamma(\alpha)} \leq c_1|y^*|.\end{aligned}$$

For the bound for I_2 we begin as above and see that

$$|I_2| \leq L|y(\xi) - y(x^*)|\frac{1}{\Gamma(\alpha+1)}\epsilon^\alpha.$$

We then continue in a slightly different way, using the Mean Value Theorem of differential calculus, to derive

$$\begin{aligned}|I_2| &\leq L\Omega|\xi - x^*|\frac{1}{\Gamma(\alpha+1)}\epsilon^\alpha \leq L\Omega\frac{1}{\Gamma(\alpha+1)}\epsilon^{\alpha+1} \\ &\leq L\Omega\frac{1}{\Gamma(\alpha+1)}[\epsilon^*]^{\alpha+1} \leq L\Omega\frac{1}{\Gamma(\alpha+1)}[M_2(\alpha)]^{\alpha+1} = c_2|y^*|.\end{aligned}$$

The final steps are then once again as in the first case. ∎

Using the argumentation of the proof of Theorem 1.4, we can reformulate Theorem 2.1 in terms of Caputo-type fractional differential equations.

THEOREM 2.2. *Let $f : [0,b] \times [c,d] \to \mathbb{R}$ be continuous and satisfy a Lipschitz condition with respect to the second variable. Consider two solutions y_1 and y_2 of the differential equation*

$$D^\alpha_{*0} y_j(x) = f(x, y_j(x)) \qquad (j = 1,2)$$

subject to the initial conditions $y_j(0) = y_{j0}$, respectively, where $y_{10} \neq y_{20}$. Moreover, assume one of the following two conditions:

(a) There exist two constants $c_1, c_2 > 0$ such that $c_1 + c_2 < 1$ and $c_2|y_{20} - y_{10}| \geq L\Omega$ where Ω is a positive constant that satisfies $\omega(y_1 - y_2; h) \leq \Omega h^\alpha$ for all h.

(b) $f \in C^1([0,b] \times [c,d])$.

Then, there exists some $\alpha_0 > 0$ such that for all $\alpha \in (0, \alpha_0)$ and for all x where both $y_1(x)$ and $y_2(x)$ exist, we have $y_1(x) \neq y_2(x)$.

References

[1] H. Brunner, A. Pedas and G. Vainikko, The piecewise polynomial collocation method for nonlinear weakly singular Volterra equations. *Math. Comp.* **68** (1999), 1079-1095.

[2] M. Caputo, Linear models of dissipation whose Q is almost frequency independent–II. *Geophys. J. Royal Astronom. Soc.* **13** (1967), 529-539; reprinted in *Fract. Calc. Appl. Anal.* **11**, No 1 (2008), 4-14.

[3] C. Corduneanu, *Principles of Differential and Integral Equations.* 2nd ed., Chelsea Publ. Comp., New York (1977).

[4] K. Diethelm, N. J. Ford, Analysis of fractional differential equations. *J. Math. Anal. Appl.* **265** (2002), 229-248.

[5] R. Gorenflo, F. Mainardi, Fractional oscillations and Mittag-Leffler functions. In: *Proceedings of the International Workshop on the Recent Advances in Applied Mathematics (RAAM '96)*, Kuwait University, Dept. of Mathematics and Computer Science, Kuwait (1996), 193-208.

[6] S. G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional Integrals and Derivatives: Theory and Applications.* Gordon and Breach, Yverdon (1993).

Institut Computational Mathematics
Technische Universität Braunschweig
Pockelsstr. 14
38106 Braunschweig – GERMANY
e-mail: k.diethelm@tu-bs.de

Received: March 10, 2008

and

GNS Gesellschaft für Numerische Simulation mbH
Am Gaußberg 2
38114 Braunschweig – GERMANY
e-mail: diethelm@gns-mbh.com